Finite Mathematics

Summation Notation: given a collection of numbers $\{a_1, a_2, a_3, \cdots, a_n\}$, the symbol $\sum_{k=1}^{n} a_k$ represents their sum, that is

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

$\sum$: The greek capital letter Σ (sigma) indicates a sum.

a_k: The symbol a_k represents the kth term.

k: The letter k is called the index of summation or the summation variables.

1 and n: indicate the extremes values (or limits) of the summation variable. [1: lower limit, n: upper limit]

To illustrate how sigma notation works, consider the sum: $1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2+10^2$ in which each term is of the form k^2, where k is one of the integers from 1 to 10, in sigma notation this sum can be written as $\sum_{k=1}^{10} k^2$ which is read "The summation of k^2, where k runs from 1 to 10".

In general, if $f(k)$ is a function of k, and a and b are integers s.t. $a \le b$, then $\sum_{k=a}^{k=b} f(k)$ denotes the sum of the terms that result when we substitute successive integers for k, starting with $k=a$ and ending with $k=b$.

Ex $\sum_{k=4}^{k=8} k^3 = 4^3+5^3+6^3+7^3+8^3$

$$\sum_{k=1}^{6} (3k+2) = (3\cdot1+2)+(3\cdot2+2)+(3\cdot3+2)+(3\cdot4+2)$$

$$(3\cdot 5+2)+(3\cdot 6+2) = 5+8+11+14+17+20$$
$$= 75$$

$$\sum_{k=1}^{4} k^2(k-3) = 1^2(1-3)+2^2(2-3)+3^2(3-3)+4^2(4-3)$$
$$= (-2)+(-4)+0+16 = 10$$

$$\sum_{k=0}^{4} \frac{1}{k+1} = \frac{1}{0+1}+\frac{1}{1+1}+\frac{1}{2+1}+\frac{1}{3+1}+\frac{1}{4+1} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$$

$$\sum_{k=1}^{5} (-1)^{k+1}(2k) = (-1)^{1+1}(2\cdot 1)+(-1)^{2+1}(2\cdot 2)+(-1)^{3+1}(2\cdot 3)+(-1)^{4+1}(2\cdot 4)$$
$$+(-1)^{5+1}(2\cdot 5)$$

$$= 2-4+6-8+10 = 6$$

$$\sum_{k=1}^{5} 7 = 7+7+7+7+7 = 5\cdot 7 = 35$$

In general $\sum_{k=1}^{n} c = nc$ $\quad c$ constant

$$\sum_{k=0}^{3} k\cdot \sin\left(\frac{k\pi}{3}\right) = 0\cdot\sin 0+1\cdot\sin\left(\frac{\pi}{3}\right)+2\sin\frac{2\pi}{3}+3\sin\frac{3\pi}{3}$$

Changing the index of summation

Ex Express $\sum_{k=3}^{7} 5^{k-2}$ in sigma notation so that the lower limit of summation is 0 rather than 3.

Solution: define a new summation index j by means of the formula $j = k-3 \Rightarrow k = j+3$

$$\therefore \sum_{k=3}^{7} 5^{k-2} = \sum_{j=0}^{4} 5^{(j+3)-2} \Rightarrow \begin{cases} \text{when } k=3 \Rightarrow j=0 \\ k=7 \Rightarrow j=4 \end{cases}$$

$$= \sum_{j=0}^{4} 5^{j+1}$$

$\therefore \sum_{j=0}^{4} 5^{j+1}$ and $\sum_{k=3}^{7} 5^{k-2}$ both denote

the sum $5 + 5^2 + 5^3 + 5^4 + 5^5$.

A general sum with n terms might be written as

$$a_1 + a_2 + a_3 + \cdots + a_n$$

or in sigma notation as

$$\sum_{k=1}^{n} a_k \quad \text{or} \quad \sum_{j=1}^{j=n} a_j \quad \text{or} \quad \sum_{m=1}^{m=n} a_m .$$

Properties of Sigma notation:

Theorem a) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

b) $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

c) $\sum_{k=1}^{n} c a_k = c \sum_{k=1}^{n} a_k$

proof of (a) $\sum_{k=1}^{n} (a_k + b_k) = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdots$

$\cdots + (a_n + b_n)$

$= (a_1 + a_2 + a_3 + \cdots + a_n) + (b_1 + b_2 + b_3 + \cdots + b_n)$

$= \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$.

proof of (c) $\sum_{k=1}^{n} c a_k = c a_1 + c a_2 + c a_3 + \cdots + c a_n$

$= c (a_1 + a_2 + a_3 + \cdots + a_n)$

$= c \sum_{k=1}^{n} a_k$.

Summation formulas:

Theorem a. $\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

b. $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Hint: start with $\sum_{k=1}^{n} (k+1)^4 - k^4$

c. $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$.

proof of (a) If we write the terms of

$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$ — (1)

in the opposite order

$$\sum_{k=1}^{n} k = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1 \quad \text{—— (2)}$$

Adding (1) and (2) term by term yields

$$2\sum_{k=1}^{n} k = \underbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1)}_{n \text{ terms}} = n(n+1)$$

Thus $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

$(a+b)^3 = a^3 + b^3 + 3ab(a+b)$

proof of (b)

We know that

$$(k+1)^3 - k^3 = k^3 + 3k^2 + 3k + 1 - k^3 = 3k^2 + 3k + 1$$

we obtain:

$$\sum_{k=1}^{n} (k+1)^3 - k^3 = \sum_{k=1}^{n} (3k^2 + 3k + 1) \quad \text{—— (3)}$$

Writing out the left side of (3)

$$[2^3 - 1^3] + [3^3 - 2^3] + [4^3 - 3^3] + \cdots + [(n+1)^3 - n^3] = -1 + (n+1)^3$$

The right side of (3) is $\sum_{k=1}^{n} (3k^2 + 3k + 1) = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$

$\therefore$ we get $-1 + (n+1)^3 = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$

$$\sum_{k=1}^{n} 1 = \underbrace{1 + 1 + 1 + \cdots + 1}_{n \text{ terms}} = n$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{from (a)}$$

$$\therefore \quad -1 + (n+1)^3 = 3\sum_{k=1}^{n} k^2 + 3\,\frac{n(n+1)}{2} + n$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}\left[(n+1)^3 - 3\,\frac{n(n+1)}{2} - (n+1)\right] * \frac{2}{2}$$

$$= \frac{n+1}{6}\left[2(n+1)^2 - 3n - 2\right]$$

$$= \frac{n+1}{6}\left[2(n^2+2n+1) - 3n - 2\right] = \frac{n+1}{6}(2n^2+n)$$

$$= \frac{n(n+1)(2n+1)}{6}$$

Ex. Evaluate $\sum_{k=1}^{30} k(k+1)$

Solution:

$$\sum_{k=1}^{30} k(k+1) = \sum_{k=1}^{30} (k^2+k) = \sum_{k=1}^{30} k^2 + \sum_{k=1}^{30} k$$

$$= \frac{30(31)(61)}{6} + \frac{30(31)}{2} = 9920$$

Remark: In formulas such as

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

or $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

the left side in open form and the right side in closed form.

Ex. Express $\sum_{k=1}^{n} (3+k)^2$ in closed form.

Solution:

$$\sum_{k=1}^{n} (3+k)^2 = \sum_{k=1}^{n} (9+6k+k^2) = \sum_{k=1}^{n} 9 + 6\sum_{k=1}^{n} k + \sum_{k=1}^{n} k^2$$

$$= 9n + 6\frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1}{3}n^3 + \frac{7}{2}n^2 + \frac{23}{6}n.$$

يتبع (خطوات توضيح البسط)

توضيح تفاصيل الشكل الأخير

$$9n + 6\frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6}$$

$$= 9n + 3n^2 + 3n + \frac{(n^2+n)(2n+1)}{6}$$

$$= 3n^2 + 12n + \frac{2n^3 + n^2 + 2n^2 + n}{6}$$

$$= \frac{18n^2 + 72n + 2n^3 + n^2 + 2n^2 + n}{6}$$

$$= \frac{2n^3}{6} + \frac{21n^2}{6} + \frac{23n}{6}$$

$$= \frac{1}{3}n^3 + \frac{7}{2}n^2 + \frac{23}{6}n .$$

Mathematical Induction:

Let $P(n)$ denotes the statement $(xy)^n = x^n y^n$ where x and y are real numbers. Thus $P(1)$ represents the statement $(xy)^1 = x^1 y^1$, $P(2)$ denotes $(xy)^2 = x^2 y^2$, $P(3)$ is $(xy)^3 = x^3 y^3$, and so on. It is easy to show that $P(1)$, $P(2)$ and $P(3)$ are true statements.

Since the set of positive integers is infinite, it is impossible to check the validity of $P(n)$ for every positive integer n, in order to give a proof, the method below is required.

Principle of mathematical induction:

If with each positive integer n there is associated a statement $P(n)$, then all the statements $P(n)$ are true provided the following two conditions hold:

i) $P(1)$ is true

ii) whenever k is a positive integer such that $P(k)$ is true, then $P(k+1)$ is also true.

Ex1 prove that the sum of the first n positive odd integers is n^2. (statement $P(n)$)

or show that $1+3+5+\cdots+(2n-1) = n^2$.

Solution: Let $P(n)$ denotes the statement

$$1+3+5+\cdots+(2n-1) = n^2$$

i) when $n=1$; left side $2(1)-1 = 2-1 = 1$

Right side $1^2 = 1$

$\therefore$ left side = Right side

$\therefore$ $P(1)$ is true.

(ii) assume that $p(k)$ is true (i.e. at $n=k$)

that is: $1+3+5+\cdots+(2k-1) = k^2$ ——①

Show that $p(k+1)$ is true (i.e. at $n=k+1$)

that is:

$$\underbrace{1+3+5+\cdots+(2k-1)}_{k^2 \text{ (from ①)}} + (2(k+1)-1) = (k+1)^2$$

$$\Rightarrow k^2 + (2(k+1)-1) = (k+1)^2$$

$$k^2 + 2k + 2 - 1 = k^2 + 2k + 1$$

$$k^2 + 2k + 1 = k^2 + 2k + 1$$

$\therefore p(k+1)$ is true.

$\therefore$ The statement $p(n)$ is true for all n.

Ex 2 prove that for any positive integer n, the sum of the first n positive integers is $n(n+1)/2$.

Solution: for any positive integer n, let $p(n)$ denotes the statement: $1+2+3+\cdots+n = \frac{n(n+1)}{2}$

i) when $n=1$; left side 1

Right side $\frac{1(1+1)}{2} = \frac{2}{2} = 1$

$\therefore$ left side = Right side $\therefore p(1)$ is true.

ii) assume that the statement is true at $n=k$

that is $1+2+3+\cdots+k = \frac{k(k+1)}{2}$ ——①

Show that the statement is true at $n = k+1$

that is;

$$\underbrace{1+2+3+\cdots+k}_{\frac{k(k+1)}{2}\ \text{from}\ (1)} + k+1 = \frac{(k+1)(k+1+1)}{2}$$

$$\Rightarrow \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

$$\frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$$\frac{k^2 + k + 2k + 2}{2} = \frac{k^2 + 2k + k + 2}{2}$$

$\therefore$ $p(k+1)$ is true.

$\therefore$ The statement $p(n)$ is true for all positive integer n.

Ex 3 prove that $2^{3n} - 1$ is divisible by 7 if n is any positive integer.

Solution: i) when $n = 1$; $\Rightarrow 2^{3(1)} - 1 = 8 - 1 = 7$;

which is divisible by 7.

$\therefore$ $p(1)$ is true.

ii) assume the statement is true at $n = k$

that is: $2^{3k} - 1$ is divisible by 7

$\Rightarrow 2^{3k} - 1 = 7m$, m is some positive integer. ——— (1)

show that the statement is true at $n = k+1$

that is $2^{3(k+1)} - 1$ is divisible by 7.

$\Rightarrow 2^{(3k+3)} - 1$

$= 2^{3k} \cdot 2^{3} - 1$

$2^{3k} = (7m+1)$ from (1)

$\Rightarrow (7m+1) \cdot 2^3 - 1$

$= (7m+1)\,8 - 1$

$= 56m + 8 - 1$

$= 56m + 7 = 7(8m+1) = 7p$

let $p = 8m+1$ is positive integer.

$\therefore p(k+1)$ is divisible by 7

$\therefore$ The statement is true for all n.

Ex Show that $n^2 - n + 2$ is an even number for all $n \geq 1$.

Solution: i) when $n = 1$; $\Rightarrow 1^2 - 1 + 2 = 2$

which is an even number.

$\therefore$ the statement is true at $n = 1$.

ii). assume the statement is true at $n = k$

that is: $k^2 - k + 2$ is an even number. ___ (1)

show that the statement is true at $n = k+1$.

that is

$$(k+1)^2 - (k+1) + 2 \text{ is an even number}$$

$$\Rightarrow k^2 + 2k + 1 - k - 1 + 2$$

$$= \underbrace{(k^2 - k + 2)}_{\text{is an even number from (1)}} + 2k$$

and we know $2k$ is even number, therefore

$$\overbrace{(k^2 - k + 2)}^{\text{even}} + \overbrace{2k}^{\text{even}} \text{ is an even number for any } n \geq 1$$

∴ The statement is true for all $n \geq 1$.

Exercises:

1- Evaluate $\sum_{k=1}^{100} k$, $\sum_{k=3}^{100} k$, $\sum_{r=1}^{6} \frac{1}{r}$, $\sum_{n=2}^{3} \frac{4}{n(n+2)}$

$\sum_{n=0}^{5} 1$, $\sum_{k=1}^{6} (k - k^3)$, $\sum_{i=1}^{4} \left(\sum_{j=0}^{5} (i+j) \right)$.

2- Evaluate $\sum_{k=0}^{n} a r^k - r \sum_{k=0}^{n} a r^k$.

3- Express in sigma notation, but do not evaluate

a. $1+2+3+\cdots+10$ b. $1+2+2^2+2^3+2^4$

c. $3\cdot1 + 3\cdot2 + 3\cdot3 + \cdots + 3\cdot20$ d. $1-3+5-7+\cdots+15$

4. Which of the following are valid identities?

a. $\sum_{j=1}^{\infty} f(j) = \sum_{i=1}^{\infty} f(i)$ b. $\sum_{k=0}^{3} (2k + bk) = \sum_{k=1}^{3} b_k + q$

c. $\sum_{i=1}^{n} (a_i + c) = \sum_{i=1}^{n} a_i + nc$ d. $\sum_{i=1}^{n} \frac{a_i}{b_i} = \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$.

5. Prove ~~that~~ the given statement by induction.

a. $2 + 4 + 6 + \cdots + 2n = n(n+1)$

b. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4} n^2 (n+1)^2$

c. $6^n - 1$ is divisible by 5.

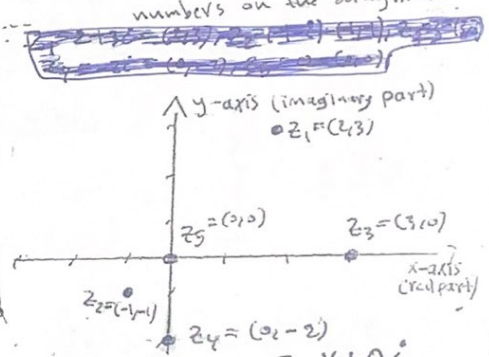
② Complex numbers:

- Defn: The complex number is defined to have the form $x+iy$ or (x,y), where x and y are real numbers and $i=\sqrt{-1}$ we shall denote to the complex number by z. That is

$z=x+iy \equiv (x,y)$, where:
x is called the real part of z and denoted by $R(z)$.
y is called the imaginary part of z and denoted by $I(z)$.
i is called the imaginary unit.

Ex(1): Find the real part and the imaginary part of the following complex numbers: $2+3i$, $-1-i$, 3, $-2i$, 0. Represent the complex numbers on the diagram.

☆ $z_1=2+3i \Rightarrow R(z_1)=2$, $I(z_1)=3$
$z_2=-1-i \Rightarrow R(z_2)=-1$, $I(z_2)=-1$
$z_3=3 \Rightarrow R(z_3)=3$, $I(z_3)=0$
$z_4=-2i \Rightarrow R(z_4)=0$, $I(z_4)=-2$
$z_5=0=0+0i \Rightarrow R(z_5)=0$, $I(z_5)=0$



- Note: Each real number $x \in \mathbb{R}$ can be written as $x+0i$, therefore, each real number is complex number.

- Defn: The absolute value or modulus of a complex numb $z=x+iy$ is defined as:

$$|z|=\sqrt{x^2+y^2}.$$

- Defn.: The conjugate of a complex number $z=x+iy$ is defined as $\bar{z}=x-iy$.

Ex(2): If $z=3-4i$. Find $\bar{z}$, $R(z)$, $I(\bar{z})$, $|z|$, $|\bar{z}|$.

☆ $z=3-4i \Rightarrow \bar{z}=3+4i$
$R(z)=3$, $I(\bar{z})=4$
$|z|=\sqrt{3^2+(-4)^2}=\sqrt{9+16}=\sqrt{25}=5=|\bar{z}|$.

Ex(3): If $z = x+iy$. Show that: (1) $|\bar{z}| = |z|$, (2) $\overline{\bar{z}} = z$.

☆ Since $z = x+iy \Rightarrow \bar{z} = x-iy$

(1) $|z| = \sqrt{x^2+y^2}$, and $|\bar{z}| = \sqrt{x^2+(-y)^2} = \sqrt{x^2+y^2} = |z|$.

(2) $z = x+iy \Rightarrow \bar{z} = x-iy \Rightarrow \overline{\bar{z}} = x-(-y)i = x+iy = z$.

Ex(4): If $z = x+iy$. Show that $|z| \geq R(z)$.

☆ $|z| = \sqrt{x^2+y^2} \geq \sqrt{x^2} = x = R(z)$.

- Some area of applications:

Consider the quadratic equation $ax^2+bx+c=0$, and we know that the solution is given by the formula: $x = \frac{-b \mp \sqrt{b^2-4ac}}{2a}$ when the discriminant

b^2-4ac $\begin{cases} >0, & \text{we say that the roots are real and different.} \\ =0, & \text{" " " " " " " " equals.} \\ <0, & \text{" " " " " " " " complex.} \end{cases}$

Ex(5): Solve for x if:

(1) $x^2+1=0$. (2) $x^2-4x+6=0$. (3) $x^3-2x-4=0$

☆

(1) $x^2+1=0 \Rightarrow x^2-i^2=0 \Rightarrow (x-i)(x+i)=0 \Rightarrow x = \mp i$.

(2) $x^2-4x+6=0 \Rightarrow x = \frac{4 \mp \sqrt{16-4\times6}}{2} = \frac{4 \mp \sqrt{-8}}{2} = \frac{4 \mp \sqrt{8i^2}}{2}$

$= \frac{4 \mp 2\sqrt{2}i}{2} = 2 \mp \sqrt{2}i$

$\therefore x = 2+\sqrt{2}i$ and $x = 2-\sqrt{2}i$.

(3) $x^3-2x-4=0 \Rightarrow (x-2)(x^2+2x+2)=0 \Rightarrow$

$x-2=0 \Rightarrow x=2$

or $x^2+2x+2=0 \Rightarrow x = \frac{-2 \mp \sqrt{4-4\times2}}{2} = \frac{-2 \mp \sqrt{-4}}{2} = \frac{-2 \mp \sqrt{4i^2}}{2}$

$= \frac{-2 \mp 2i}{2} = -1 \mp i$.

$\therefore x = -1+i \; ; -1-i \; , 2$.

Operations on Complex numbers:

(1) Two Complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal if and only if: $x_1 = x_2$ and $y_1 = y_2$, that is:

$$x_1 + iy_1 = x_2 + iy_2 \iff x_1 = x_2 \text{ and } y_1 = y_2$$

(2) Addition and Subtraction:

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then:

$$z_1 \mp z_2 = (x_1 + iy_1) \mp (x_2 + iy_2) = (x_1 \mp x_2) + i(y_1 \mp y_2)$$

(3) Multiplication:

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2) = x_1 x_2 + i x_1 y_2 + i x_2 y_1 + i^2 y_1 y_2$$

$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

(4) Scalar Multiplication:

Let $z = x + iy$, and K is any constant, then:

$$Kz = K(x + iy) = Kx + iKy.$$

(5) Division:

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then:

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2} = \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{x_2^2 - i^2 y_2^2}$$

$$= \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\right).$$

Ex(6): Let $z_1 = -1 + 2i$ and $z_2 = -3 - 2i$. Find $z_1 + z_2$, $z_1 \cdot z_2$, $\frac{z_1}{z_2}$.

- $z_1 + z_2 = (-1 + 2i) + (-3 - 2i) = (-1 - 3) + i(2 - 2) = -4 + 0i = -4$
- $z_1 \cdot z_2 = (-1 + 2i) * (-3 - 2i) = [(-1)(-3)] - (2)(-2)] + i[(-1)(-2) + (-3)(2)]$
  $= (3 + 4) + i[2 - 6) = 7 - 4i$
- $\frac{z_1}{z_2} = \frac{-1 + 2i}{-3 - 2i} \cdot \frac{-3 + 2i}{-3 + 2i} = \frac{3 - 2i - 6i - 4}{9 + 4} = \frac{-1 - 8i}{13} = \frac{-1}{13} - \frac{8}{13} i.$

Laws of Algebra:

(1) $z_1 + z_2 = z_2 + z_1$ commutative law of addition

$z_1 \cdot z_2 = z_2 \cdot z_1$ " " of multiplication

(2) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ associative law of addition

$z_1 (z_2 z_3) = (z_1 z_2) z_3$ " " of multiplication

(3) $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

$(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$ } distributive law

Proof: (1) $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$

$= (x_1 + x_2) + i(y_1 + y_2)$

$= (x_2 + x_1) + i(y_2 + y_1)$; since x_1, x_2, y_1, y_2 are real numbers.

$= (x_2 + iy_2) + (x_1 + iy_1)$

$= z_2 + z_1$.

Ex: prove that if $z_1 z_2 = 0$ then either $z_1 = 0$ or $z_2 = 0$.

Solution: $z_1 z_2 = 0$

$(x_1 + iy_1)(x_2 + iy_2) = 0$

$(x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) = 0$

$x_1 x_2 - y_1 y_2 = 0$

$x_1 y_2 + x_2 y_1 = 0$ — square and add

$x_1^2 x_2^2 + y_1^2 y_2^2 + x_1^2 y_2^2 + x_2^2 y_1^2 = 0$

$x_1^2 (x_2^2 + y_2^2) + y_1^2 (y_2^2 + x_2^2) = 0$

$(x_1^2 + y_1^2)(x_2^2 + y_2^2) = 0$

either $x_1^2 + y_1^2 = 0$

$\because x_1, y_1 \in \mathbb{R} \Rightarrow x_1 = y_1 = 0 \Rightarrow z_1 = 0$

Or $x_2^2 + y_2^2 = 0$

$\because x_2, y_2 \in \mathbb{R} \Rightarrow x_2 = y_2 = 0 \Rightarrow z_2 = 0$.

Ex(7): Let $z = x+iy$, $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$

Show that: ① $z+\bar{z} = 2R(z)$. ② $z-\bar{z} = 2i\,I(z)$.

③ $z\cdot\bar{z} = |z|^2$ ④ $\overline{z_1-z_2} = \bar{z}_1-\bar{z}_2$ ⑤ $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

Solution:

① $z+\bar{z} = (x+iy)+(x-iy) = 2x+0i = 2x = 2R(z)$

② $z-\bar{z} = (x+iy)-(x-iy) = 0+2yi = 2iy = 2i\,I(z)$.

③ $z\cdot\bar{z} = \left[x\cdot x-(y)(-y)\right]+i\left[(x)(-y)+(x)(y)\right]$

$= x^2+y^2+0i = x^2+y^2 = |z|^2$.

④ $\because z_1-z_2 = (x_1+iy_1)-(x_2+iy_2) = (x_1-x_2)+i(y_1-y_2)$

$\therefore \overline{z_1-z_2} = (x_1-x_2)-i(y_1-y_2) = (x_1-iy_1)-(x_2-iy_2)$

$= \bar{z}_1-\bar{z}_2$.

⑤ H.w.

Ex(8): Let $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$. Show that

① $|z_1\cdot z_2| = |z_1|\cdot|z_2|$ ② $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

③ $|z_1+z_2| \le |z_1|+|z_2|$

Solution:

① $|z_1\cdot z_2| = \left|(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)\right| = \sqrt{(x_1x_2-y_1y_2)^2+(x_1y_2+x_2y_1)^2}$

$= \sqrt{x_1^2x_2^2-2x_1x_2y_1y_2+y_1^2y_2^2+x_1^2y_2^2+2x_1y_2x_2y_1+x_2^2y_1^2}$

$= \sqrt{x_1^2x_2^2+x_1^2y_2^2+x_2^2y_1^2+y_1^2y_2^2}$

$= \sqrt{x_1^2(x_2^2+y_2^2)+y_1^2(x_2^2+y_2^2)} = \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$

$= \sqrt{x_1^2+y_1^2}\cdot\sqrt{x_2^2+y_2^2} = |z_1|\cdot|z_2|$.

③ $|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2}) = (z_1+z_2)(\bar{z}_1+\bar{z}_2)$

from Ex(7): ③

$= z_1\bar{z}_1+z_1\bar{z}_2+z_2\bar{z}_1+z_2\bar{z}_2$

$= z_1\bar{z}_1+z_2\bar{z}_2+z_1\bar{z}_2+z_2\bar{z}_1$

$= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2}$ from Ex(7):①

$= |z_1|^2 + |z_2|^2 + 2R(z_1\bar{z}_2)$ → where $z_2\bar{z}_1 = \overline{z_1\bar{z}_2}$ and $z_1\bar{z}_2 + \overline{z_1\bar{z}_2} = 2R(z_1\bar{z}_2)$

$\therefore |z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2R(z_1\bar{z}_2)$

since $|z_1\bar{z}_2| \overset{Ex(4)}{\geq} R(z_1\bar{z}_2)$

$\therefore |z_1+z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2|$

$\leq |z_1|^2 + |z_2|^2 + 2|z_1||\bar{z}_2|$ above $|z_2| = |\bar{z}_2|$

$\leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$\therefore |z_1+z_2|^2 \leq (|z_1| + |z_2|)^2 \Rightarrow |z_1+z_2| \leq |z_1| + |z_2|$.

Ex(9) Simplify $\dfrac{i+i^2+i^3+i^4+i^5}{1+i}$.

Solution: $\dfrac{i+i^2+i^3+i^4+i^5}{1+i} = \dfrac{i-1-i+1+i}{1+i} = \dfrac{i}{1+i}\cdot\dfrac{1-i}{1-i} = \dfrac{i-i^2}{1^2+1^2}$

$= \dfrac{i+1}{2} = \dfrac{1}{2} + \dfrac{1}{2}i$.

Ex(10) Solve for x and y if $x+iy = (3+4i)^2 - 2(x-iy)$.

Solution: $x+iy = 9+24i+16i^2-2x+2yi$

$\Rightarrow x+iy+2x-2yi = -7+24i \Rightarrow 3x-iy = -7+24i$

$\Rightarrow 3x = -7 \Rightarrow x = \dfrac{-7}{3}$

and

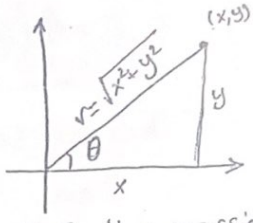
$-y = 24 \Rightarrow y = -24$.

- Polar Representation for the Complex Number:

We define the Complex number $z=x+iy$ as an ordered pair $z=(x,y)$. So, we can represent another form to a complex number called the Polar form as follows:

$$z = x+iy = (x,y)$$
$$= r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta), \text{ where}$$
$$r=\sqrt{x^2+y^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$r=|z|$ and θ is called the argument of the Complex number z and denoted by $\theta = Arg\ z$.



$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$

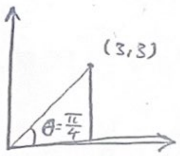
$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

Ex(11): Express the following complex numbers in Polar form:

(1) $3+3i$. (2) $-1+\sqrt{3}\,i$. (3) -1. (4) $-2-2\sqrt{3}\,i$

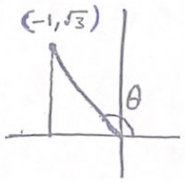
(1) $z=3+3i=(3,3)=r(\cos\theta+i\sin\theta)$

$$r=|z|=\sqrt{3^2+3^2}=\sqrt{18}=3\sqrt{2}$$
$$\tan\theta=\frac{y}{x}=\frac{3}{3}=1 \Rightarrow \theta=\frac{\pi}{4}$$
$$\therefore z=3\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$



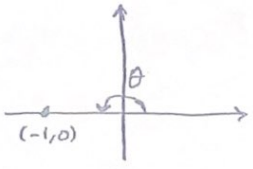
(2) $z=-1+\sqrt{3}\,i=(-1,\sqrt{3})=r(\cos\theta+i\sin\theta)$

$$r=|z|=\sqrt{(-1)^2+(\sqrt{3})^2}=\sqrt{1+3}=\sqrt{4}=2$$
$$\tan\theta=\frac{y}{x}=\frac{\sqrt{3}}{-1} \Rightarrow \theta=120=\frac{2\pi}{3}$$
$$\therefore z=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$$

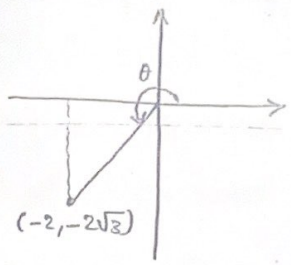


(3) $z=-1=-1+0i=(-1,0)=r(\cos\theta+i\sin\theta)$

$$r=|z|=\sqrt{(-1)^2+0^2}=\sqrt{1}=1$$
$$\tan\theta=\frac{y}{x}=\frac{0}{1}=0 \Rightarrow \theta=\pi$$
$$\therefore z=\cos\pi+i\sin\pi$$



(4) $z = -2 - 2\sqrt{3}\,i = (-2, -2\sqrt{3}) = r(\cos\theta + i\sin\theta)$



$r = |z| = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$

$\tan\theta = \frac{y}{x} = \frac{-2\sqrt{3}}{-2} = \sqrt{3} \Rightarrow \theta = 240 = \frac{4\pi}{3}$

$\therefore z = 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

- **Theorem (1):** If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then:

(1) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$. (2) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(3) $\mathrm{Arg}(z_1 \cdot z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$. (4) $\mathrm{Arg}\left(\frac{z_1}{z_2}\right) = \mathrm{Arg}(z_1) - \mathrm{Arg}(z_2)$

Proof: of (1) and (3)

(Hint): $\sin(A \mp B) = \sin A\cos B \mp \sin B\cos A$

$\cos(A \mp B) = \cos A\cos B \pm \sin A\sin B$

(1) we have $r_1 = |z_1|$, $r_2 = |z_2|$, $\theta_1 = \mathrm{Arg}(z_1)$, $\theta_2 = \mathrm{Arg}(z_2)$

$z_1 \cdot z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2 [\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_2\cos\theta_2 + i^2\sin\theta_1\sin\theta$

$= r_1 r_2 [(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)]$

$z_1 \cdot z_2 = r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

$\therefore |z_1 z_2| = r_1 r_2 = |z_1||z_2|$

$\mathrm{Arg}(z_1 \cdot z_2) = \theta_1 + \theta_2 = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$

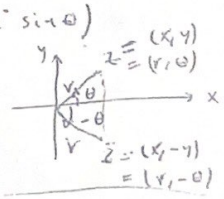
(2) $\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)} \cdot \frac{r_2(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$

Conjugate of polar form: $z = x+iy = (x,y)$, $\bar{z} = x-iy = (x,-y)$

$z = r(\cos\theta + i\sin\theta)$

$\bar{z} = r(\cos(-\theta) + i\sin(-\theta))$

$\bar{z} = r(\cos\theta - i\sin\theta)$



$= \frac{r_1}{r_2} \cdot \frac{\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2}{\cos^2\theta_2 - i^2\sin^2\theta_2}$ $\quad \cos^2\theta_2 + \sin^2\theta_2 = 1$

$= \frac{r_1}{r_2}[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$

$\therefore \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$ and $\mathrm{Arg}\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \mathrm{Arg}(z_1) - \mathrm{Arg}(z_2)$.

Ex(12): Let $z_1 = 1+i$ and $z_2 = \sqrt{3} - i$. Express $z_1 z_2$ and $\frac{z_1}{z_2}$ in polar form and find $|z_1 z_2|$, $\text{Arg}(z_1 z_2)$, $\left|\frac{z_1}{z_2}\right|$ and $\text{Arg}\left(\frac{z_1}{z_2}\right)$.

Solution: $z_1 = 1+i = (1,1) = r_1(\cos\theta_1 + i\sin\theta_1)$

$r_1 = \sqrt{2}$, $\tan\theta = \frac{1}{1} = 1 \Rightarrow \theta_1 = \frac{\pi}{4} \Rightarrow z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$z_2 = \sqrt{3} - i = (\sqrt{3}, -1) = r_2(\cos\theta_2 + i\sin\theta_2)$

$r_2 = 2$, $\tan\theta_2 = \frac{-1}{\sqrt{3}} \Rightarrow \theta_2 = 330 = \frac{11\pi}{6}$ or $\theta_2 = -30 = \frac{-\pi}{6}$

$\Rightarrow z_2 = 2\left[\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right]$ or $z_2 = 2\left[\cos\frac{-\pi}{6} + i\sin\frac{-\pi}{6}\right]$

$z_1 \cdot z_2 = r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$

$= 2\sqrt{2}\left[\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)\right] = 2\sqrt{2}\left[\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right]$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$

$= \frac{\sqrt{2}}{2}\left[\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)\right] = \frac{1}{\sqrt{2}}\left[\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right]$.

Ex(13): Find the value of $z_1 z_2$ and $|z_1 z_2|$, where $z_1 = 5(\cos 110 + i\sin 110)$ and $z_2 = 7(\cos 160 + i\sin 160)$.

Solution: $z_1 z_2 = 35\left(\cos(110+160) + i\sin(110+160)\right)$

$= 35(\cos 270 + i\sin 270) = 35(0 - i) = -35i$

$|z_1 z_2| = r_1 r_2 = 35$.

Theorem (2): "DeMoiver's Theorem":

If n is a positive integer, then

$$z^n = \left[r(\cos\theta + i\sin\theta)\right]^n = r^n\left(\cos n\theta + i\sin n\theta\right). \quad \text{---} \quad (*)$$

Proof: By Mathematical induction:

i) when $n = 1$; lft side $= \left[r(\cos\theta + i\sin\theta)\right]^1$

Right side $= r^1(\cos\theta + i\sin\theta)$

$\therefore$ Left side = Right side

$\therefore$ p(1) is true.

ii) assume eq. (*) is true for $n=k$ that is

$$\left[r(\cos\theta+i\sin\theta)\right]^k = r^k(\cos k\theta + i\sin k\theta) \quad \text{——— (1)}$$

show that eq(*) is true for $n=k+1$, that is:

$$\left[r(\cos\theta+i\sin\theta)\right]^{k+1} = r^{k+1}\left(\cos(k+1)\theta + i\sin(k+1)\theta\right)$$

$$\underbrace{\left[r(\cos\theta+i\sin\theta)\right]^k}_{r^k(\cos k\theta + i\sin k\theta)\ \text{from (1)}} \cdot \left[r(\cos\theta+i\sin\theta)\right]^1 = r^{k+1}\left(\cos(k+1)\theta + i\sin(k+1)\theta\right)$$

$$\therefore r^k(\cos k\theta + i\sin k\theta)\cdot r(\cos\theta+i\sin\theta) = r^k\cdot r\left[\cos(k+1)\theta + i\sin(k+1)\theta\right]$$

$$\Rightarrow r^k\cdot r\left(\cos(k\theta+\theta) + i\sin(k\theta+\theta)\right) = r^k\cdot r\left[\left(\cos(k\theta+\theta) + i\sin(k\theta+\theta)\right)\right]$$

$\therefore$ eq. (*) is true for $n=k+1$.

$$\therefore z^n = \left[r(\cos\theta+i\sin\theta)\right]^n = r^n\left[\cos n\theta + i\sin n\theta\right]$$

Ex (14) Use DeMoiver's theorem to calculate the following:

(1) $(1+i)^{12}$, (2) $(\sqrt{3}+i)^5$.

Solution: (1) $z = 1+i \Rightarrow x=1, y=1 \Rightarrow r = \sqrt{1^2+1^2} = \sqrt{2}$

and $\tan\theta = \frac{1}{1} = 1 \Rightarrow \theta = \frac{\pi}{4}$

$$\therefore z = r(\cos\theta+i\sin\theta) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\therefore z^{12} = (1+i)^{12} = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{12}$$

$$= (\sqrt{2})^{12}\left[\cos 12\frac{\pi}{4} + i\sin 12\frac{\pi}{4}\right]$$

$= 2^6 \left[\cos 3\pi + i \sin 3\pi\right] = 64\left[\cos \pi + i \sin \pi\right] = 64\left[-1 + i(0)\right]$

$= -64.$

(2) $z = \sqrt{3} + i \Rightarrow x = \sqrt{3}$ and $y = 1$

$\Rightarrow r = \sqrt{(\sqrt{3})^2 + 1^2} = 2$ and $\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}.$

$\therefore z = r(\cos\theta + i\sin\theta) = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$\therefore z^5 = (\sqrt{3} + i)^5 = \left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^5$

$= 2^5\left[\cos 5\frac{\pi}{6} + i\sin 5\frac{\pi}{6}\right]$

$\therefore z^5 = 32\left[\frac{-\sqrt{3}}{2} + i\frac{1}{2}\right] = -16\sqrt{3} + 16i.$

Exercises

(1) Show that

(a) $2(5-3i)-3(-2+i)+5(i-3) = 1-4i$

(b) $(3-2i)^3 = -9-46i$

(c) $\frac{5}{3-4i} + \frac{10}{4+3i} = \frac{11}{5} - \frac{2}{5}i$

(d) $\frac{(1+i)(2+3i)(4-2i)}{(1+2i)^2(1-i)} = \frac{16}{5} - \frac{2}{5}i$

(e) $\left|\frac{2-4i}{5-7i}\right| = \left(\frac{10}{37}\right)^{1/2}$

(f) $\left|\frac{1}{1+3i} - \frac{1}{1-3i}\right| = \frac{3}{5}$

(2) Solve for x if

(a) $x^2+2x+2=0$, Ans: $x = -1 \pm i$

(b) $2x^3-x^2-x+2=0$, Ans: $x = -1, \frac{3}{4} \pm \frac{\sqrt{7}}{4}i$

(c) $2x^4-3x^3-7x^2-8x+6=0$, Ans: $x = 3, \frac{1}{2}, -1 \pm i$

(d) $x^4+2x^3+3x^2+2x+2=0$, Ans: $x = \pm i, -1 \pm i$

(e) $x^4-2x^3+3x^2=0$, Ans: $x = 0, 0, \frac{1}{2} \pm \frac{1}{\sqrt{2}}i$

(3) Solve for x and y if

(a) $(3-2i)(x+iy) = 2(x-2iy)+2i-1$, Ans: $x=-1, y=0$

(b) $\left(\frac{1+i}{1-i}\right)^2 + \frac{1}{x+iy} = 1+i$, Ans: $x = \frac{2}{5}, y = -\frac{1}{5}$

(4) If $z = x+iy$. Show that $\frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

(5) Show that $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^9 = -1$ DeMoivre

(6) If $\frac{x-iy}{x+iy} = a+ib$. Show that $a^2+b^2=1$

(7) Express the following Complex no.'s in Polar form

(a) $3-3i$ (b) -4 (c) i (d) $-6i$ (e) $-1+i$ (f) $\frac{1}{2}+\frac{\sqrt{3}}{2}i$

(g) $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ (h) $-2+i$ (i) $(1-\sqrt{3})+(1+\sqrt{3})i$ (j) $\frac{1+\sqrt{3}i}{1-\sqrt{3}i}$, (k) $(1+\sqrt{3}i)^6$

(8) Express the following complex numbers in Cartesian form

(a) $4\sqrt{3}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ (b) $\frac{3}{2}(\cos 45+i\sin 45)$

(c) $6(\cos 270+i\sin 270)$ (d) $-2(\cos\pi+i\sin\pi)$

(9) Simplify the following

(a) $(-\sqrt{3}+i)^7$ Ans: $64(\sqrt{3}-1)$

(b) $\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)^{40}$, $[2(\cos 25+i\sin 25)][5(\cos 110+i\sin 110)]$

(10) If $z=1+i$. Find z^2, z^3, z^4, z^6, z^8

(11) Simplify $\left[\frac{(1+\sin\theta)+i\cos\theta}{(1+\sin\theta)-i\cos\theta}\right]^n$, $\frac{12(\cos 16+i\sin 16)}{[3(\cos 44+i\sin 44)][2\cos 62+i\sin 62)]}$

(12) If n is +ive integer. show that

$(\sqrt{3}+i)^n+(\sqrt{3}-i)^n = 2^{n+1}\cos\frac{n\pi}{6}$

(13) If $z=\cos\theta+i\sin\theta$ and $1+z+z^2+\cdots+z^{n-1}=\frac{1-z^n}{1-z}$, $z\neq 1$

show that

(a) $1+2\cos\theta+2\cos 2\theta+\cdots+2\cos(n-1)\theta = \frac{\sin(n-\frac{1}{2})\theta}{\sin\frac{\theta}{2}}$

(b) $\sin\theta+\sin 2\theta+\cdots+\sin(n-1)\theta = \frac{\cos\frac{\theta}{2}-\cos(n-\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}$

(14) Express $\sin 4\theta$ and $\cos 4\theta$ in terms of $\sin\theta$ and $\cos\theta$

(15) Prove that

(a) $\arg(z^n) = \arg(z)+2\pi n$

(b) $\arg\left(\prod_{i=1}^{n} z_i\right) = \sum_{i=1}^{n}\arg(z_i)$

(16) Find the roots of the following: $z^3=1$, $z^2=i$, $z^3=-8i$, $z^6=64$.

③ Polynomials:

- Defn.: The polynomial equation and polynomial function of degree n are defined by the algebric forms:

$$a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_nx^n=0 \quad , a_n\neq 0$$

and $f(x)=a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_nx^n \quad , a_n\neq 0$,

where $a_0, a_1, a_2, \cdots, a_n$ are constants. For examples

$-4x^3+\frac{1}{2}x-\sqrt{3}=0$, is a polynomial equation of degree 3

$f(x)=2x^6+4x^5-3x^2-2$, is a polynomial function of degree 6.

- Properties:

(1) Equal polynomials: Two polynomials are said to be equal if their corresponding coefficients are equal. That is:

$a_0+a_1x+a_2x^2+\cdots+a_nx^n=b_0+b_1x+b_2x^2+\cdots+b_nx^n$, if

$a_0=b_0$, $a_1=b_1$, $a_2=b_2$, $\cdots$, $a_n=b_n$

Ex(1): Find a, b, c, d if $2x^3+3x^2+x-1=(ax+b)(x^2+2x+2)+cx+d$.

$\therefore$ $2x^3+3x^2+x-1=(ax+b)(x^2+2x+2)+cx+d$

$=ax^3+2ax^2+2ax+bx^2+2bx+2b+cx+d$

$=ax^3+(2a+b)x^2+(2a+2b+c)x+(2b+d)$

$\therefore$ $\boxed{a=2}$; $2a+b=3 \Rightarrow 4+b=3 \Rightarrow \boxed{b=-1}$; $2a+2b+c=1 \Rightarrow 4-2+c=$

$\Rightarrow \boxed{c=-1}$; $2b+d=-1 \Rightarrow -2+d=-1 \Rightarrow \boxed{d=1}$

(2) Addition, Subtruction and Multiplication:

If $f(x)$ and $g(x)$ are two polynomials of degree n and m respectively, with $n \geq m$, then

i- $f(x) \mp g(x)$ is a polynomial of degree $\leq n$.

ii- $f(x) \cdot g(x)$ is a polynomial of degree $m+n$.

Ex(2): If $f(x)=2x^2+x+1$ and $g(x)=2x^2-1$. Find $f(x)+g(x)$, $f(x)-g(x)$, $f(x)\cdot g(x)$.

$\therefore$ $f(x)+g(x)=2x^2+x+1+2x^2-1=4x^2+x$, is a poly. of degree 2.

$f(x)+g(x)=2x^2+x+1-2x^2+1=x+2$, is a poly. of degree 1.

$f(x)\cdot g(x)=(2x^2+x+1)(2x^2-1)=4x^4+2x^3-x-1$, is a poly. of degree 4

(3) Quotient of two polynomials:

If $f(x)$ and $g(x)$ are polynomials of degree n and m with $n \geq m$ then there exist two polynomials $p(x)$ and $r(x)$ such that, either $r(x)=0$, or ~~both $p(x)$ and~~ $r(x)$ of degree $<m$, and

$$\frac{f(x)}{g(x)}=p(x)+\frac{r(x)}{g(x)} \quad \text{or} \quad f(x)=p(x)g(x)+r(x).$$

Ex(3): If $f(x)=4x^3-2x^2+4x+3$ and $g(x)=2x^2+x-1$. Find two polynomials $p(x)$ and $r(x)$, such that: $f(x)=p(x)\,g(x)+r(x)$.

$\therefore$ $f(x)-2x\,g(x)=4x^3-2x^2+4x+3-2x(2x^2+x-1)=-4x^2+6x+3$

$f(x)-2x\,g(x)+2g(x)=-4x^2+6x+3+2(2x^2+x-1)=8x+1$

$\therefore$ $f(x)=2x\,g(x)-2g(x)+8x+1=(2x-2)g(x)+(8x+1)$

$\therefore$ $p(x)=2x-2$, $r(x)=8x+1$

Note: This method is identical of finding $\frac{f(x)}{g(x)}$ by long division:

$$\therefore \frac{4x^3-2x^2+4x+3}{2x^2+x-1}=(2x-2)+\frac{8x+1}{2x^2+x-1}$$

$$\therefore \frac{f(x)}{g(x)}=(2x-2)+\frac{8x+1}{g(x)}$$

$$\Rightarrow f(x)=(2x-2)\,g(x)+(8x+1)$$

$$\begin{array}{r|l} & 2x-2 \\ \hline 2x^2+x-1 & 4x^3-2x^2+4x+3 \\ & \mp 4x^3 \mp 2x^2 \pm 2x \\ \hline & -4x^2+6x+3 \\ & \pm 4x^2 \pm 8x \mp 2 \\ \hline & 8x+1 \end{array}$$

Ex(4): Find two polynomials $p(x)$ and $r(x)$, s.t: $f(x)=p(x)g(x)+r(x)$, where $f(x)=2x^5-3x^4+4x^3+2x^2+x-3$, and $g(x)=x^2+x+1$.

☆

$$\begin{array}{r|l} & 2x^3-5x^2+7x \\ x^2+x+1 & 2x^5-3x^4+4x^3+2x^2+x-3 \\ & \mp 2x^5 \mp 2x^4 \mp 2x^3 \\ \hline & -5x^4+2x^3+2x^2+x-3 \\ & \pm 5x^4 \pm 5x^3 \pm 5x^2 \\ \hline & 7x^3+7x^2+x-3 \\ & \mp 7x^3 \mp 7x^2 \mp 7x \\ \hline & -6x-3 \end{array}$$

$\therefore$ $p(x)=2x^3-5x^2+7x$

and

$r(x)=-6x-3$

Note: When $g=x-a$, there is a quick method of finding $p(x)$ and $r(x)$, such that $f(x)=p(x)g(x)+r(x)$.

Ex(5): If $f(x)=x^5-4x^3+2x^2-5x-8$ and $g(x)=x-3$. Find two polynomials, s.t: $f(x)=p(x)g(x)+r(x)$.

☆

$f(x)-x^4g(x)=x^5-4x^3+2x^2-5x-8-x^4(x-3)=3x^4-4x^3+2x^2-5x-8$

$f(x)-x^4g(x)-3x^3g(x)=3x^4-4x^3+2x^2-5x-8-3x^3(x-3)=5x^3+2x^2-5x-8$

$f(x)-x^4g(x)-3x^3g(x)-5x^2g(x)=5x^3+2x^2-5x-8-5x^2(x-3)=17x^2-5x-8$

$f(x)-x^4g(x)-3x^3g(x)-5x^2g(x)-17xg(x)=17x^2-5x-8-17x(x-3)=46x-8$

$f(x)-x^4g(x)-3x^3g(x)-5x^2g(x)-17xg(x)-46g(x)=46x-8-46(x-3)=130$

$\therefore f(x)=(x^4+3x^3+5x^2+17x+46)g(x)+130 \Rightarrow$

$p(x)=x^4+3x^3+5x^2+17x+46$, and $r(x)=130$.

or The coefficients of $f(x)$:

	1	0	−4	2	−5	−8
3		3	9	15	51	138
	1	3	5	17	46	130

$\therefore f(x)=\underbrace{(x^4+3x^3+5x^2+17x+46)}_{p(x)}g(x)+\underbrace{130}_{r(x)}$

Ex(6): Let $f(x) = x^4+4x^3+4x^2+7x+13$, and $g(x) = x+2$. Find two polynomials $p(x)$ and $r(x)$, s.t: $f(x) = p(x)g(x)+r(x)$.

	1	4	4	7	13
-2		-2	-4	0	-14
	1	2	0	7	$\boxed{-1}$

$\therefore f(x) = \underbrace{(x^3+2x^2+7)}_{p(x)}g(x)\underbrace{-1}_{r(x)}$

$\therefore p(x) = x^3+2x^2+7$, and $r(x) = -1$

- Roots of a polynomial Equation:

Theorem (1): "The Fundamental theorem in Algebra"

Every polynomial equation of degree $n \geq 1$ has at least one real (or imaginary) root

For examples: (1) The equation:

$2x^6+15x^5+3x^4-94x^3+84x^2+15x-25=0$, is a polynomial of degree 6, which can be written as:

$$(2x+1)(x+5)^2(x-1)^3 = 0.$$

We can see that each of $(2x+1)$, $(x+5)$ and $(x-1)$ is a factor of the equation So we say that $-\frac{1}{2}, -5, -5, 1, 1, 1$ are the roots of the equation and $-\frac{1}{2}$ is simple root (or single root), -5 is doublicated root (or repeated 2 times) and 1 is triple root (or repeated 3 times). So it is clear that the equation has 6 roots.

Note: A value of x which satisfies the equation $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$ is called a root or solution of the equation and also a zero of the polynomial.

The factor theorem and its converse:

If r is a root of the polynomial equation $f(x)=0$ then $x-r$ is a factor of $f(x)$; and conversely if $x-r$ is a factor of $f(x)$, then r is a root of the equation $f(x)=0$.

Ex: Show by the factor theorem that $x+3$ is a factor of: $f(x) \equiv x^3 + x^2 - 7x - 3 = 0$.

Solu: we want to show that -3 is a root of

$$x^3 + x^2 - 7x - 3 = 0 .$$

Substituting $x = -3$, we obtain

$$(-3)^3 + (-3)^2 - 7(-3) - 3 = -27 + 9 + 21 - 3 = 0$$

$\therefore$ The equation has -3 as a root $\Rightarrow$ $x+3$ is a factor of $x^3 + x^2 - 7x - 3$.

Theorem (2): Every polynomial of degree $n \geq 1$ has exactly n roots.

Proof:

Let $f_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, be a polynomial of degree n. Then from theorem (1), $f_n(x) = 0$ has at least one root, say α_1, that implies $(x - \alpha_1)$ is a factor of $f_n(x)$, and:

$f_n(x) = (x - \alpha_1) f_{n-1}(x)$, where $f_{n-1}(x)$ is a polynomial of degree $n-1$.

Also, the equation $f_{n-1}(x) = 0$, has at least one root, say α_2 and $f_{n-1}(x) = (x - \alpha_2) f_{n-2}(x)$, where $f_{n-2}(x)$ is a polynomial of degree $n-2$. Therefore:

$$f_n(x) = (x - \alpha_1)(x - \alpha_2) f_{n-2}(x).$$

If we continue the same process n times, we find

$f_n(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \cdots (x - \alpha_n) f_0(x)$, where $f_0(x)$ is a polynomial of degree zero, that is $f_0(x) = a_0 x^0 = a_0$

Therefore: $f_n(x) = a_0 (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$, and the roots of the polynomial equation $f_n(x)$ are $\alpha_1, \alpha_2, \cdots, \alpha_n$, and they are exactly n roots.

Ex. The equation $(x-1)(x+2)(x-5)=0$

is of degree 3 and it has three

distinct (different) roots: $1, -2, 5$.

Ex. The equation $(x-4)^2(x-3)^4(x+2)=0$

is of degree 7 and it has seven

roots:

The roots; 4 as a double root,

3 as a root of multiplicity four,

-2 as a simple root (or single root).

The total number of roots is seven.

Ex. Form a cubic equation $f(x)=0$ whose roots are:

$2, -3, \frac{1}{5}$.

Sol. $2, -3, \frac{1}{5}$ are roots $\Rightarrow (x-2), (x+3), (x-\frac{1}{5})$ are factors of $f(x)$.

$\Rightarrow (x-2)(x+3)(x-\frac{1}{5})=0$

$\Rightarrow 5x^3+4x^2-31x+6=0$ is the equation

Theorem (3): If $\alpha = a+ib$ is a root for the polynomial equation $f(x)=0$, then $\bar{\alpha} = a-ib$ is also a root.

Proof: Since α is a root of the equation $f(x)=0$, then $(x-\alpha)$ is a factor of the equation $f(x)=0$. We shall prove that if $(x-\bar{\alpha})$ is a factor of the equation $f(x)=$ then $\bar{\alpha}$ is a root of $f(x)=0$.

Let $g(x) = (x-\alpha)(x-\bar{\alpha}) = x^2 - 2(\alpha+\bar{\alpha})x + \alpha\bar{\alpha}$.

Now if we prove that $g(x)$ is a factor of $f(x)=0$, then $(x-\bar{\alpha})$ is also a factor of $f(x)=0$.

We divide $f(x)$ on $g(x)$ and suppose the result is

$$\frac{f(x)}{g(x)} = p(x) + \frac{r(x)}{g(x)} \rightarrow f(x) = p(x)\,g(x) + r(x) \quad \text{--- (1)}$$

where $r(x)$ degree is less than the degree of $g(x)$, say $r(x) = cx+d$.

Since α is a root of equation $f(x)=0 \Rightarrow f(\alpha)=0$

and $(x-\alpha)$ is a factor of $g(x) \Rightarrow g(\alpha)=0$.

Substitute α in eq.(1) to get

$c\alpha+d=0 \Rightarrow c(a+ib)+d=0 \Rightarrow ca+icb+d=0 \Rightarrow$

$(ca+d)+cbi = 0+0i \Rightarrow ca+d=0$ and $cb=0$

Since α is a complex number, then $b\neq 0$, $\therefore c=0$

and $ca+d=0 \Rightarrow d=0$

$\therefore f(x) = p(x)\,g(x) = p(x)[(x-\alpha)(x-\bar{\alpha})]$

$\therefore x-\bar{\alpha}$ is a factor of $f(x)=0 \Rightarrow$

$\bar{\alpha}$ is a root of $f(x)=0$.

Let $z = a+ib$ $\therefore f(z)$ is a poly in z with real coefficients.

$f(z)$ can be written in the form $u+iv$

u, v are real no's.

Hence $f(a+ib) = u+iv$

$f(a-ib) = u-iv$

$a+ib$ is a root of $f(z)=0$

$\therefore f(a+ib) = u+iv = 0 \Rightarrow u = v = 0$

$\therefore u-iv = f(a-ib) = 0$

$\therefore a-ib$ is a root of $f(z)=0$.

Ex: Solve the equation $x^4 - 4x^3 + 10x^2 + 12x - 39 = 0$

if $2-3i$ is one of the roots.

Solu: since $2-3i$ is a root

$\therefore$ $2+3i$ is also a root by Theorem (3).

$\therefore$ $x-(2-3i)$ and $x-(2+3i)$ are factors

of $f(x)=0$ $\Rightarrow f_n(x) = (x-(2-3i))(x-(2+3i)) \cdot f_{n-2}(x)$

Then the product $\therefore f_{n-2}(x) = \frac{f_n(x)}{(x-(2-3i))(x-(2+3i))}$

$$[x-(2-3i)][x-(2+3i)] = [(x-2)+3i][(x-2)-3i]$$

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$$= (x-2)^2 - (3i)^2 = x^2 - 4x + 13.$$

The other factors obtained by Long division:

$$\frac{x^4 - 4x^3 + 10x^2 + 12x - 39}{x^2 - 4x + 13} = x^2 - 3$$

the other factors

$$\begin{array}{r|l} x^2 - 4x + 13 & x^4 - 4x^3 + 10x^2 + 12x - 39 \\ & \mp x^4 \pm 4x^3 + 13x^2 \\ \hline & -3x^2 + 12x - 39 \\ & \mp 3x^2 \pm 12x \mp 39 \\ \hline & 0 \quad 0 \quad 0 \end{array}$$

The roots of $x^2 - 3 = 0$ are $\mp\sqrt{3}$.

Hence the solutions set is

$$\{2-3i, 2+3i, -\sqrt{3}, \sqrt{3}\}.$$

Ex: Find the equation whose roots are $-1, -1, 3+2i$.

Solu: since $3+2i$ is a root, then $3-2i$ is also a root, then: the equation is;

$$(x+1)(x+1)(x-(3+2i))(x-(3-2i)) = 0$$

$$\Rightarrow (x+1)^2 (x^2 - 6x + 13) = 0 \Rightarrow (x^2+2x+1)(x^2-6x+13) = 0$$

$$\Rightarrow x^4 - 4x^3 + 2x^2 + 20x + 13 = 0 \quad \text{is the equation.}$$

Ex: Find ~~an equation~~ the solution of the equation

$$x(x-7)^3(x^2+8x+97)=0.$$

Sol. $x(x-7)^3(x^2+8x+97)=0$

$x=0 \Rightarrow r_1=0$ جذر مكرر

$x=7 \Rightarrow r_2=7, r_3=7, r_4=7.$

$x^2+8x+97=0 \Rightarrow x_{1,2}=\dfrac{-8\mp\sqrt{64-4(97)}}{2}$

$$=\frac{-8\mp 2\sqrt{16-97}}{2}$$

$$=-4\mp\sqrt{-81}=-4\mp 9i$$

$\therefore r_5=-4-9i, \quad r_6=-4+9i.$

H.W. Find an equation whose roots are

$3+2i, \ \frac{4}{5}, \ -1, \ -1.$

Ex(5): Find the equation whose roots are $-1, -1, 3+2i$

* Since $3+2i$ is a root, then $3-2i$ is also a root, then:

the equation is:

$(x+1)(x+1)(x-(3+2i))(x-(3-2i))=0 \Rightarrow$

$(x+1)^2(x^2-6x+13)=0 \Rightarrow (x^2+2x+1)(x^2-6x+13)=0 \Rightarrow$

$x^4-4x^3+2x^2+20x+13=0$ is the equation.

- Upper and Lower Bounds for the real roots of the polynomial equation:

Let $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0$, $a_n \neq 0$ ... ①

be a polynomial equation of degree n. If $a_n > 0$, we divide $f(x)$ by $(x-K)$, $K = 1, 2, 3, \ldots$, That is, we find $\frac{f(x)}{x-K}$. Whenever the coefficients result are all positive or zero, then $K=a$, $a = 1, 2, 3, \ldots$ is the upper bound for the positive real roots of equation (1). To find the lower bound for the negative real roots of eq(1), we set $x=-y$, then eq(1) become

$$a_n y^n - a_{n-1} y^{n-1} + \cdots + a_2 y^2 + a_1 y + (-1)^n a_0 = 0 \quad \cdots ②$$

Similarly, if $K=a$, $a=1,2,3,\ldots$ is the upper bound for the positive real roots of eq(2), then $K=-a$, is the lower bound for the negative roots of eq(1).

Ex(6): Find the upper and lower bounds for the roots of the equation: $2x^3+x^2-16=0$ ---①

For K=1

	2	1	0	-16
1		2	3	3
	2	3	3	-13

not all positive or zero!

For K=2

	2	1	0	-16
2		4	10	20
	2	5	10	4

all positive

∴ K=2 is the upper bound for the positive real roots of eq.①

Set $x=-x$, we have

$$-2x^3+x^2-16=0 \Rightarrow 2x^3-x^2+16=0 \quad ---②$$

For K=1

	2	-1	0	16
1		2	1	1
	2	1	1	17

all positive

∴ K=1 is the upper bound for the real roots of eq.(2)

⇒ K=-1 is the lower bound of eq.(1)

Ex(7): Find the upper and lower bounds for the roots of the following polynomials:

(1) $2x^3-9x^2+5x+7=0$. (2) $x^5-3x^4-6x+10=0$

(1) $2x^3-9x^2+5x+7=0$ ---①

Let K=1 ⇒

	2	-9	5	7
1		2	-7	-2
	2	-7	-2	5

Let K=2 ⇒

	2	-9	5	7
2		4	-10	-10
	2	-5	-5	-3

Let K=3 ⇒

	2	-9	5	7
3		6	-9	-12
	2	-3	-4	-5

Let K=5

	2	-9	5	7
5		10	5	50
	2	1	10	57

$\therefore K=5$ is the upper bound of the real roots of eq.(1)

-set $x=-x \Rightarrow 2(-x)^3-9(-x)^2+5(-x)+7=0 \Rightarrow$

$-2x^3-9x^2-5x+7=0 \Rightarrow$

$2x^3+9x^2+5x-7=0$ ---②

Let $K=1 \Rightarrow$

	2	9	5	-7
1		2	11	16
	2	11	16	9

$\therefore K=1$ is the upper bound of eq.(2)

Therefore, $K=-1$ is the lower bound of the real roots of eq.(1)

(2) $x^5-3x^4-6x+10=0$ --- ①

Let $K=1$

	1	-3	0	0	-6	10
1		1	-2	-2	-2	-8
	1	-2	-2	-2	-8	2

Let $K=2$

	1	-3	0	0	-6	10
2		2	-2	-4	-8	-28
	1	-1	-2	-4	-14	-18

Let $K=3$

	1	-3	0	0	-6	10
3		3	0	0	0	-18
	1	0	0	0	-6	-8

Let $K=4$

	1	-3	0	0	-6	10
4		4	4	16	64	232
	1	1	4	16	58	242

$\therefore K=4$ is the upper bound of the real roots of eq.(1)

Set $x=-x \Rightarrow -x^5-3x^4+6x+10=0 \Rightarrow$

$x^5+3x^4-6x-10=0$ ---②

Let $K=1$

	1	3	0	0	-6	-10
1		1	4	4	4	-2
	1	4	4	4	-2	-12

Let $K=2$

	1	3	0	0	-6	-10
2		2	10	20	40	68
	1	5	10	20	34	58

$\therefore K=2$ is the upper bound of eq.(2),

Therefore

$K=-2$ is the lower bound of the real roots of eq.(1)

- Relation Between the roots and Coefficients:

- Polynomial of degree 2 has 2 roots, say α_1 and α_2, then

$$x^2+ax+b=(x-\alpha_1)(x-\alpha_2)$$
$$=x^2-(\alpha_1+\alpha_2)x+\alpha_1\alpha_2$$

$\therefore a=-(\alpha_1+\alpha_2)$, $b=\alpha_1\alpha_2$

- Polynomial of degree 3 has 3 roots, say α_1, α_2 and α_3, th

$$x^3+ax^2+bx+c=(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$$
$$=x^3-(\alpha_1+\alpha_2+\alpha_3)x^2+(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)x-\alpha_1\alpha_2\alpha_3$$

$\therefore a=-(\alpha_1+\alpha_2+\alpha_3)$, $b=\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3$, $c=-\alpha_1\alpha_2\alpha_3$

- Polynomial of degree 4 has 4 roots, say $\alpha_1, \alpha_2, \alpha_3$ and α_4, the

$$x^4+ax^3+bx^2+cx+d=(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)$$
$$=x^4-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)x^3+(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_1\alpha_4+\alpha_2\alpha_3+\alpha_2\alpha_4+\alpha_3\alpha_4)x^2$$
$$-(\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_3\alpha_4+\alpha_2\alpha_3\alpha_4)x+\alpha_1\alpha_2\alpha_3\alpha_4$$

$\therefore a=-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$, $b=\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_1\alpha_4+\alpha_2\alpha_3+\alpha_2\alpha_4+\alpha_3\alpha_4$,

$c=-(\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_3\alpha_4+\alpha_2\alpha_3\alpha_4)$, $d=\alpha_1\alpha_2\alpha_3\alpha_4$

and so on

Ex(8): Find the roots of the equation $x^3-3x^2+4x-2=0$, if the product of two roots is 2.

* Let α, β, γ be the roots of the eqn. with $\alpha\beta=2$

$\therefore x^3-3x^2+4x-2=(x-\alpha)(x-\beta)(x-\gamma)$
$=x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\alpha\gamma+\beta\gamma)x-\alpha\beta\gamma$

$\therefore \alpha+\beta+\gamma=3$ ---①

$\alpha\beta+\alpha\gamma+\beta\gamma=4$ ---②

$\alpha\beta\gamma=2$ ---③

$\alpha\beta=2$ ---④

From (3) and (4), we have $2\gamma = 2 \Rightarrow \gamma = 1$

From (1), we have $\alpha+\beta+1=3 \Rightarrow \alpha+\beta=2$ and from (4).

we have $\beta = \frac{2}{\alpha} \Rightarrow \alpha + \frac{2}{\alpha} = 2 \Rightarrow \alpha^2+2=2\alpha \Rightarrow \alpha^2-2\alpha+2=0$

$\therefore \alpha = \frac{2 \mp \sqrt{4-(4)(1)(2)}}{2} = \frac{2 \mp \sqrt{-4}}{2} = \frac{2 \mp 2i}{2} = 1 \mp i$

$\therefore$ The roots are: $1, 1+i, 1-i$

Ex(9): Find the cubic Sum of the squared roots of the equation $x^4 - 4x^3 + 3x^2 + x - 1 = 0$ --- (*)

We require $(\alpha^2+\beta^2+\gamma^2+\delta^2)^3$, assuming that $\alpha, \beta, \gamma, \delta$ are the roots of eq(*). Therefore, $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$ are the factors of eq(*).

$$x^4 - 4x^3 + 3x^2 + x - 1 = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$$
$$= x^4 - (\alpha+\beta+\gamma+\delta)x^3 + (\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)x^2$$
$$- (\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)x + \alpha\beta\gamma\delta$$

$\therefore \alpha+\beta+\gamma+\delta = 4$ -- (1)

$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = 3$ --- (2)

$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -1$ --- (3)

$\alpha\beta\gamma\delta = -1$ --- (4)

Now:
$$(\alpha+\beta+\gamma+\delta)^2 = \alpha^2+\beta^2+\gamma^2+\delta^2+2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$$

$$\Rightarrow 4^2 = \alpha^2+\beta^2+\gamma^2+\delta^2+2(3) \Rightarrow \alpha^2+\beta^2+\gamma^2+\delta^2 = 16-6 = 10$$

$\therefore (\alpha^2+\beta^2+\gamma^2+\delta^2)^3 = 10^3 = 1000$.

Ex(10): If α, β, γ are the roots of the equation: $x^3 - 6x^2 - 3x + a = 0$. Find the value of a, and solve the equation if $\alpha = 2\beta + 2\gamma$.

$$x^3 - 6x^2 - 3x + a = (x-\alpha)(x-\beta)(x-\gamma)$$

$$x^3 - 6x^2 - 3x + a = x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\alpha\gamma+\beta\gamma)x - \alpha\beta\gamma$$

$\therefore \alpha+\beta+\gamma = 6$ --- ①

$\alpha\beta+\alpha\gamma+\beta\gamma = -3$ --- ②

$\alpha\beta\gamma = -a$ --- ③

$\alpha = 2\beta + 2\gamma$ --- ④

From eq(1): $\alpha+\beta+\gamma = 6 \Rightarrow 2\alpha+2\beta+2\gamma = 12 \Rightarrow 2\alpha+\alpha = 12$

$\Rightarrow \alpha = \frac{12}{3} \Rightarrow \boxed{\alpha = 4}$

$\therefore$ eq(1) becomes: $4+\beta+\gamma = 6 \Rightarrow \beta+\gamma = 2$

From eq(2): $\alpha\beta+\alpha\gamma+\beta\gamma = -3 \Rightarrow \alpha(\beta+\gamma)+\beta\gamma = -3$

$\Rightarrow 4(2)+\beta\gamma = -3 \Rightarrow \beta\gamma = -3-8 \Rightarrow \beta\gamma = -11$

From eq(3): $\alpha\beta\gamma = -a \Rightarrow 4(-11) = -a \Rightarrow \boxed{a = 44}$

Exercises

(1) Find a, b, c, d (e) if $x^3 - 2x^2 + x - 6 = a(x-1)^2(x^2+1) + bx(x-1)(x^2+1) + cx(x^2+1) + (dx+e)x(x-1)^2$.

(2) Find two polynomials $p(x)$ and $r(x)$ such that $f(x) = p(x)\,g(x) + r(x)$ where

(a) $f(x) = 2x^5 - 3x^4 + 4x^3 + 2x^2 + x - 3$, $g(x) = x^2 + x + 1$

(b) $f(x) = x^4 + 4x^3 + 4x^2 + 7x + 13$, $g(x) = x + 2$

(c) $f(x) = x^5 - 5x^4 + 6x^3 - 8x^2 - 31x - 21$, $g(x) = x - 3$

(3) Find the polynomial equation whose roots are :-

(a) $0, 3, 3, -2$

(b) $0, 0, -1, \frac{1}{3}, \frac{2}{7}$

(c) $-2, \sqrt{11}, -\sqrt{11}$

(d) $1, \frac{1}{2} + 3i, \frac{1}{2} - 3i$

(e) $6, -1+i$

(f) $i, 3-i$

$\sum = \frac{n}{n+1} \quad \frac{\frac{k+1}{k+2}}{\frac{(k+1)+1}{k+2}} = \frac{k+1}{k+2}$

(4) Find the roots of the following equations

(a) $x(x-7)^3(x^2+8x+97) = 0$

(b) $(x-1)(x+5)^2(x^2-6x+2) = 0$

(c) $(x+4)(7x-3)^2(x^2+25) = 0$

(d) $(x+8)(2x-1)(x^2+10x+29) = 0$

(e) $(3x^2-7x)(x^2-12x+3) = 0$

(5) If the irrational number $\alpha_1 = a + \sqrt{b}$ is a root for the eq. $f(x) = c$ show that $\alpha_2 = a - \sqrt{b}$ is also a root for the eq. $f(x) = 0$, where a and b are +ive integers.

(6) If $f(0) \neq 0$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are roots for the polynomial eq. $f(x) = 0$. Show that

$$f(x) = f(0)\left(1 - \frac{x}{\alpha_1}\right)\left(1 - \frac{x}{\alpha_2}\right)\left(1 - \frac{x}{\alpha_3}\right)\cdots\left(1 - \frac{x}{\alpha_n}\right)$$

EX Show by the factor theorem that x^3+x^2-7x-3 is divisible by $x+3$.

Solution want to show -3 is a root of $x^3+x^2-7x-3=0$?

substituting $x=-3$, we obtain

$(-3)^3+(-3)^2-7(-3)-3=-27+9+21-3=0$

The equation has -3 as a root $\Rightarrow x+3$ is a factor of x^3+x^2-7x-3.

EX Form a cubic equation $f(x)=0$ whos roots are, 2, -3, $\frac{1}{5}$

Solution $2, -3, \frac{1}{5}$ are roots $\Rightarrow x-2, x+3, x-\frac{1}{5}$ are factors of $f(x) \Rightarrow (x-2)(x+3)(x-\frac{1}{5})=0$

$\Rightarrow 5x^3+4x^2-31x+6=0$ is the equation.

Synthetic Division:

EX Divide $x^5-4x^3+2x^2-5x-8$ by $x-3$

Solution تكتب في اول سطر معاملات $f(x)$. وعلى الجانب الايمن العدد r وهي تساوي 3 وتضع الناتج أسفل ثاني معامل وتجمع هذا الناتج مع هذا المعامل وحاصل الجمع تضربه مرة اخرى في 3 وتضع الناتج أسفل المعامل الثالث وتجمعها وهكذا تكون العدد الأخيرة هي الباقي

(1)	0	-4	2	-5	-8	3
	3	9	15	51	138	
(1)	3	5	17	46	130	

Remainder الباقي

$f(3)=130$

IT $f(x)=(x-r)Q(x)+R$

dividend = divisor × quotient + remainder

$x^5-4x^3+2x^2-5x-8=(x-r)(x^4+3x^3+5x^2+17x+46)+130$

EX Divide $2x^3+x^2-18x+7$ by $x-2$

Solution

(2)	1	-18	7	-2
	-4	6	+24	
(2)	-3	-12	+31	

$f(-2)=31$

$(2x^3+x^2-18x+7)=(x-2)(2x^2-3x-12)+31$

Dividend Divisor Quotient Remainder

Applications of Linear Function: Chapter 4

- Linear function: is a function whose graph produces a line. The Linear function is a polynomial of degree one that can be presented in the form: $f(x)=a+mx$ or $f(x)=mx+a$, where a is the initial starting value of the function (when $x=0$), and m is the constant rate of change of the function.

Ex(1): A taximeter is set to charge 3.30$ as a drop fee. After the initial charge, the taximeter will add 2.40$ for each mile the taxi drives. Find the model function for this taximeter then find the total charge if the taxi is drived for 7 miles.

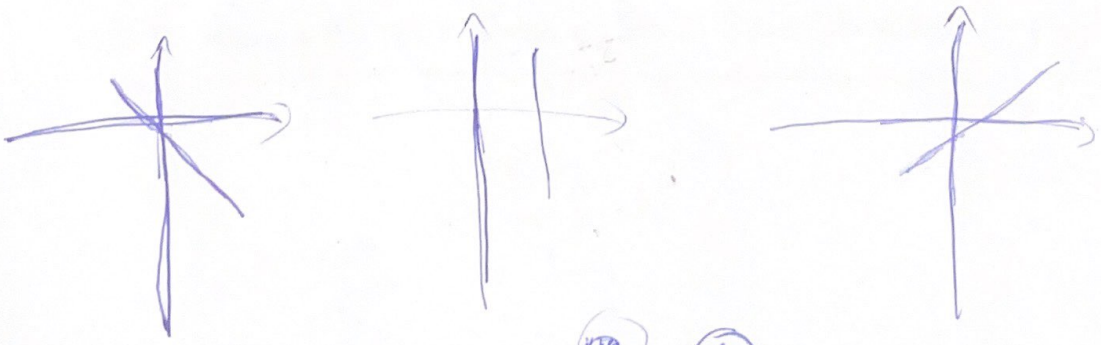
* In this example, the total taxi fare depends on the number of miles driven by the taxi. Then we will use the variable n for the numbers of miles and C for the cost in US dollar as a function of n: $C(n)$.

The initial charging value is $C(0)=3.30$.
Since 2.40 is the added charge for each mile driven, then

$$C(n)=a+mn=3.30+2.40n$$

∴ The total cost for 7 miles is:

$$C(7)=3.30+2.40(7)=20.10\$$$



The Slope, Increasing and Decreasing of functions:

The Slope of the function is the rate of change. The constant m represents the constant rate of change (also called the slop
The slope determines if the function is increasing or decreasi function.

$f(x) = a + mx$, is an increasing function if $m > 0$

$f(x) = a + mx$, is a decreasing function if $m < 0$

If $m = 0 \Rightarrow f(x) = a + 0x = a$, $f(x)$ is a constant function represented by a horizontal line passing through the point $(0, a)$

Ex(2): Jak currently owns 200 books in his library. Every month he adds 15 new books. Write a formula for the number of books, N, in his library as a number of months, m. How many books will Jak have in a year?

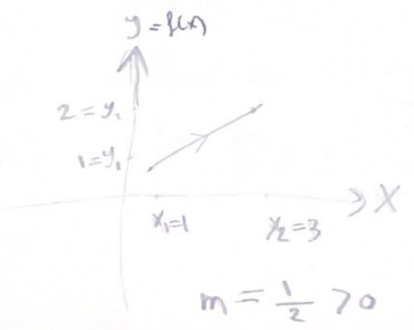
* The initial value of this model is 200 Since Jak currently owns 200 books, So $N(0) = 200$.

The number of books increases by 15 books per month, So the rate of change $m = 15$. Therefore we can write the formula:

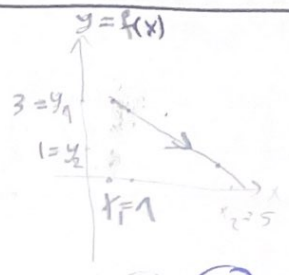
$$N(m) = 200 + 15m$$

The number of books Jak will own in a year is:

$$N(12) = 200 + 15(12) = 200 + 180 = 380 \text{ books.}$$

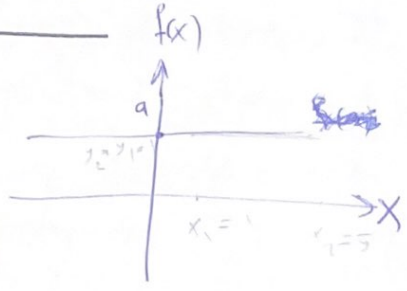


$m = \frac{1}{2} > 0$



(2)

$m = \frac{-2}{+4} = \frac{-1}{2} < 0$



$m = 0$

Ex(3): If an engineer earns 30,000 $ per year, and spends 29,000$ a year. Write an equation for the amount of money this engineer will save after y years, if he starts with nothing. How much he saves in 9 years?

* The initial amount of money the engineer starts with is 0 then a=0.

The engineer saves 1000 $ per year, then the rate of chang is 1000. Therefore, the formula can be written as:

$S(y) = 0 + 1000y = 1000y$

The amount of money this engineer will save in 9 years is

$S(9) = 1000(9) = 9000\$$.

Ex(4): The preasure P, in pounds per squared inch (PSI) on a diver depends on the depth of the diver under the water surface d, in feets. Find the preasure on the diver who dives 5 feets under the water surface, knowing that the preasure on the surface is 14.696 PSI, and that the preasure increases by 0.434 PSI each feet the diver dives under the water surface.

* The initial preasure on the diver (at the water surface, when d=0 is 14.696. The rate of change constant m=0.434.

Then the general formula for the preasure is:

$p(d) = a + md = 14.696 + 0.434d$

Therefore, the preasure on a diver dives ~~and~~ in feets under the water surface is:

$p(5) = 14.696 + 0.434(5) = 16.866$ PSI.

Calculating the rate of Change:

The rate of change describes how the output quantity changes in related to the input quantity.

Given two input values x_1 and x_2 with two corresponding output values y_1 and y_2, "(x_1, y_1) and (x_2, y_2)". The rate of change is the total change of the function (output values) divided by the change of the input values:

$$m = \frac{\text{change in output}}{\text{change in input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

This is also called the slope of the function.

Note: In function notation, $y_1 = f(x_1)$ and $y_2 = f(x_2)$, then:

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Ex(5): The population of a city increased from 23,400 to 27,800 people between 2002 and 2006. Find the rate of change for the population during this time Span.

** The rate of change is the change in population number to the change in time.

The population increased by: $27{,}800 - 23{,}400 = 4400$ peop

and the change in time is: $2006 - 2002 = 4$ years.

$\therefore$ The rate of change: $m = \frac{\Delta Pop}{\Delta time} = \frac{4400 \text{ Peo}}{4 \text{ years}} = 1100 \frac{\text{Peo}}{\text{year}}$

$= 1100$ people per year.

Notice that we know that the population was increased, so we would expect our value of m to be positive.

Ex(6): If $f(x)$ is a linear function and $f(3)=-2$ and $f(8)=1$. Find the rate of change. Then find the function formula.

☆ Since $f(3)=-2$ and $f(8)=1$, then we have:

$x_1=3,\ x_2=8,\ y_1=-2,\ y_2=1$. or $(3,-2)$ and $(8,1)$

$$\therefore m=\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}=\frac{1-(-2)}{8-3}=\frac{3}{5}=0.6$$

or take: $x_1=8,\ x_2=3,\ y_1=1,\ y_2=-2$. i.e: $(8,1)$ and $(3,-2)$.

$$\therefore m=\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}=\frac{-2-1}{3-8}=\frac{-3}{-5}=\frac{3}{5}=0.6$$

Note that it is not important which pair of values comes first

To find the function formula:

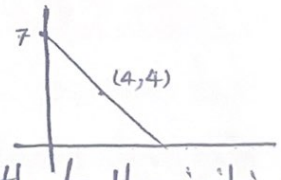
Since $f(3)=-2$ or $f(8)=1$

$\therefore f(3)=a+\frac{3}{5}(3)$ $\quad\therefore f(8)=a+\frac{3}{5}(8)$

$\Rightarrow -2=a+\frac{9}{5}\Rightarrow a=\frac{-19}{5}$ $\quad\Rightarrow 1=a+\frac{3}{5}(8)\Rightarrow a=1-\frac{24}{5}=\frac{-19}{5}$

$\therefore$ The formula of the function is: $f(x)=\frac{-19}{5}+\frac{3}{5}x$

Ex(7): Write the formula for the linear function graphed as:



☆ Looking at the graph, we can notice that the line passes through the points $(0,7)$ and $(4,4)$.

From the first value $(0,7)$ we can determine that the initial value of $f(0)=7\Rightarrow a=7$.

Now, to calculate the rate of change:

$$m=\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}=\frac{4-7}{4-0}=\frac{-3}{4}=-0.75$$

Therefore, the formula is:

$$f(x)=a+mx=7-\frac{3}{4}x$$

Note that since $m<0$, then $f(x)$ is a decreasing function

---

8): Working as an insurance salesperson, John earns a base salary and a commision on each new policy, so John's weekly income, I, depends on the number of new policies n he sells during the week. Last week he sold 3 new policies and earned 760 $ this week. the week before, he sold 5 new policies and earned 920 $. Find the equation I(n)

The given information gives us the input-output pairs: (3, 760) and (5, 920). We start by finding the rate of change constant:

$$m = \frac{\Delta y}{\Delta x} = \frac{\Delta I}{\Delta n} = \frac{I_2 - I_1}{n_2 - n_1} = \frac{920 - 760}{5 - 3} = \frac{160}{2} = 80$$

Now, to find the initial constant a, or the base salary, we have:

$$I(n) = a + mn = a + 80n$$

Since $I(3) = 760 \Rightarrow I(3) = 760 = a + 80(3) \Rightarrow a = 760 - 240 = 520\$$

Therefore, the equation I(n) can be presented as:

$$I(n) = 520 + 80n.$$

which means that John's base salary is 520$ per week and he earns 80$ as a commision for each new policy he sells.

Ex(9): Given the bellow table. Write a linear equation that represent the table.

w: no. of weeks:	0	2	4	6
p(w): no. of rats:	1000	1080	1160	1240

We can see from the table that the initial value is 1000

To calculate the rate of change, take any two rows (columns) from the given table to get:

$m = \frac{\Delta y}{\Delta x} = \frac{\Delta P}{\Delta w} = \frac{P_2 - P_1}{w_2 - w_1} = \frac{1080 - 1000}{2 - 0} = \frac{80}{2} = 40$ rats per week.

∴ The format of the linear equation is:

$P(w) = a + mw = 1000 + 40w$

Graphs of the linear function:

When plotting a linear function, there are ways to graph it:

1- By plotting points (at least two) and drawing a line through these points.

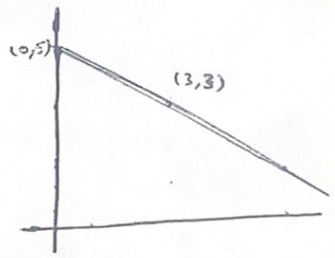
2- Using the initial point and the slope.

Ex(10): Graph the linear function: $f(x) = 5 - \frac{2}{3}x$ by plotting points.

$f(0) = 5 - \frac{2}{3}(0) = 5$,

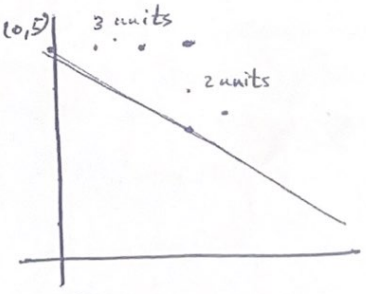
$f(3) = 5 - \frac{2}{3}(3) = 5 - 2 = 3$,

$f(6) = 5 - \frac{2}{3}(6) = 5 - 4 = 1$



Ex(11): Graph the linear function $f(x) = 5 - \frac{2}{3}x$, by using the initial point and the slope.

we can get from the equation that the initial value at $x=0$ is $f(0)=5$, $(0,5)$ and the slope $m = \frac{-2}{3}$. This tells as for every 3 units the graph runs in the horizontal, the vertical rise decreases by 2 units.



Ex (12): Graph the linear functions: 1. $f(x) = 2x$

2. $f(x) = 5 - \frac{1}{4}x$

3. $f(x) = 3$

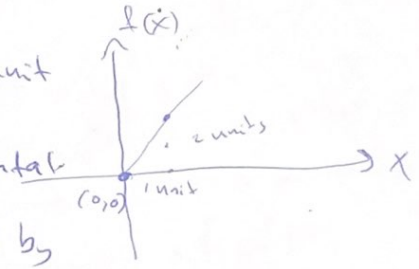
by using the initial point and the slope.

Sol:

1. From the equation $\Rightarrow a = 0$

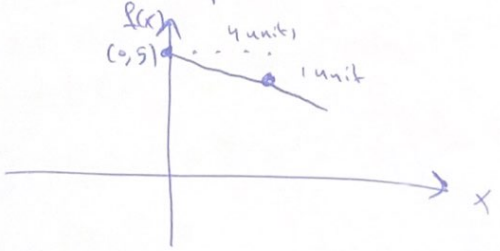
$m = 2$

This tells: for every 1 unit the graph runs in the horizontal, the vertical rise increases by 2 units.



2. From the eq. $\Rightarrow a = 5$

$m = -\frac{1}{4}$



3. $f(x) = 3$

$\Rightarrow a = 3$

$m = 0$



Solution of Non-Linear equations

الحلول العددية للمعادلات اللاخطية

1. Descarte's rule of signs

عدة ديكارت للأشارات

A polynomial $f(x)$ with the terms written in the order of descending powers of x is said to have a variation of sign if two consecutive terms have opposite signs. Any missing power of x to be disregarded (a zero coefficient is not supplied) in this definition.

تنازلية

تغير في الاشارة

For example, the polynomial $2x^5 - x^4 - 3x^3 + x^2 - 2$ has three variations of sign.

ما ثلاث تغيرات للاشارات

من $2x^5$ الى $-x^4$ ومن $-3x^3$ الى x^2 ومن x^2 الى -2

هذه التغيرات في الاشارات تعطي بعض المعلومات عن جذور معادلة متعددة الحدود التي تكون معاملاتها اعداد حقيقية

Theorem Descarte's rule of signs. the number of positive roots of a polynomial equation $f(x) = 0$ with real coefficients is equal to the number of variation of sign in $f(x)$ or is less than the number by an even integer.

the number of negative roots of $f(x) = 0$ is equal to the number of variation of sign in $f(-x)$ or less than that number by an even integer.

EX Apply Descarte's rule of signs to the eq.

$$x^3 + 7x^2 + 8x - 17 = 0$$

Solution the left member $f(x)$ has one variation of signs

Hence the equation has exactly one positive root

اختبار الجذور السالبة بدل $-x$ بـ x في المعادلة

$$f(-x) = \underbrace{-x^3}_{\text{تغير 1}} + \underbrace{7x^2}_{\text{تغير 2}} - 8x - 17$$

there are two variation of signs.

احتمالات:

either two negative roots

or no negative root.

نستنتج أن المعادلة الأصلية لها جذر موجب واحد والجذران لباقيان أما سالبان أو خياليان

The original equation has one positive root and that the two remaining roots are either both negative or both imaginary, that is, the possibilities are ⓐ one positive root and two negative roots
ⓑ one positive root and two imaginary roots.

EX Determine the information which Descarte rule provides for the equation $x^4+3x^3+4x^2+3x+5=0$

Solution there is no variation of sign in the left number
لا يوجد تغير في اشارات المعادلة

∴ the equation has no positive root.
اختبار الجذور السالبة بدل $-x$ بـ x نحصل على

$$f(-x)=\underbrace{x^4}_{\text{واحد}}\underbrace{-2x^3}_{\text{اثنين}}\underbrace{+4x^2}_{\text{ثلاثة}}\underbrace{-3x}_{\text{أربعة}}+5$$

اربعة تغيرات في الاشارة يعني أربعة جذور

four variation of sign

possibilities are
ⓐ all roots negative ⓑ all roots imaginary
ⓒ two roots negative and two roots imagin

2. Horner method for elimination roots of Polynomial equ

طريقة هورنر لحذف الجذور من معادلة متعددة الحدود

الطريقة تكون قسمة المعادلة على الجذور كما في synthetic Divison P.2

EX Two roots of the equation $2x^5-6x^4+8x^3+4x^2-20x-8=0$ are $x=2$ and $x=-1$, eliminate the two roots and find the remaining equation

Solution نقسم المعادلة على 2 ونستخدم synthetic division

	2	−6	8	4	−20	−8
2		4	−4	8	24	8
	2	−2	4	12	4	0

$\therefore p(2) = 0 \Rightarrow 2x^4 - 2x^3 + 4x^2 + 12x + 4 = 0$

	2	−2	4	12	4
−1		−2	4	−8	−4
	2	−4	8	4	0

$\therefore p(-1) = 0 \Rightarrow 2x^3 - 4x^2 + 8x + 4 = 0$

Ex Find the value of $p(-1)$ using Horner method if $f(x) = 7x^4 - 2x^3 + 3x + 8$

Sol

	7	−2	0	3	8
−1		−7	9	−9	6
	7	−9	9	−6	14

The quotient $= 7x^3 - 9x^2 + 9x - 6$ and the remainder is 14.